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DIVISIBILITY CONDITIONS IN COMMUTATIVE RINGS WITH ZERODIVISORS

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ABSTRACT

Let R be a commutative ring. In this paper, we give several divisibility and ring-theoretic conditions for R or $T(R)$ to be either zero-dimensional or von Neumann regular. We also consider divisibility conditions related to R being completely integrally closed and study several closedness conditions which hold with respect to units of $T(R)$.

INTRODUCTION

In this paper, we continue our investigation begun in Ref. [1–3, 4] of extending ring-theoretic properties in integral domains to the context of commutative rings with zerodivisors by replacing conditions on elements of the total quotient ring $T(R)$ with internal divisibility conditions on elements

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of the ring R. In the first section, we consider divisibility conditions on R which are equivalent to R or $T(R)$ being either zero-dimensional or von Neumann regular. In the second section, we give some additional conditions on R for $T(R)$ to be either zero-dimensional or von Neumann regular and relate this to ideas used in, Refs. [1,2] for subrings of a direct product of integral domains. The two main results (Theorems 2.2 and 2.3) are that $T(R)$ is zero-dimensional (resp., von Neumann regular) if and only if for each $x \in R$, there is a $y \in R$ such that $xy \in \mathrm{nil}(R)$ (resp., $xy = 0$) and $x + y$ is a regular element of R. In the third section, we investigate divisibility conditions related to R being completely integrally closed. In the fourth section, we consider several "closedness" properties which hold with respect to units of $T(R)$ and answer some questions raised in Refs. [1,2]. We show in Theorem 4.2 that if R is a Marot ring, then R satisfies these "closedness" properties with respect to $T(R)$ if and only if R satisfies them with respect to units of $T(R)$. In the final section, we briefly consider some conditions related to seminormality from Ref. [1].

Throughout, R is a commutative ring with $1 \neq 0$, group of units $U(R), \mathrm{nil}(R)$ its set of nilpotent elements, $Z(R)$ its set of zerodivisors, $\mathrm{Spec}(R)$ its set of prime ideals, and total quotient ring $T(R) = R_S$, where $S = R - Z(R)$. As usual, an $x \in R - Z(R)$ is called a *regular element* of R. When we write $x/y \in T(R)$, we will always mean that $x, y \in R$ with y a regular element of R. For any undefined terminology, see Refs. [5,6] or [7]. For an excellent survey of recent work on zero-dimensional commutative rings, see Ref. [8].

1 DIVISIBILITY CONDITIONS AND VON NEUMANN REGULAR RINGS

In this section, we give several "divisibility" conditions for a commutative ring R to be either von Neumann regular or a total quotient ring. Recall that a commutative ring R is *von Neumann regular* if for each $x \in R$, there is a $y \in R$ such that $x = x^2y$; equivalently, R is reduced and zero-dimensional.[6, Theorem 3.1] Of course, the most obvious such divisibility condition is that R is von Neumann regular if and only if $x^2 \mid x$ for all $x \in R$. Our first result gives several less obvious divisibility conditions on R which are equivalent to R being von Neumann regular (the equivalence of conditions (4)–(6) is well known).

Proposition 1.1. *The following statements are equivalent for a commutative ring R.*

(1) *Let $m \geq 2$ be a fixed integer. If $x \mid y^m$ for $x, y \in R$, then $x \mid y$.*

(2) *Let $x, y \in R$. If $x \mid y^n$ for some integer $n \geq 1$, then $x \mid y$.*
(3) *Let $x, y \in R$. If $y^n = xd$ for some integer $n \geq 1$ with $d \in R$ a nonunit, then $x \mid y$.*
(4) *All ideals of R are radical ideals.*
(5) *All principal ideals of R are radical ideals.*
(6) *R is von Neumann regular.*

Proof. (1) $\Rightarrow$ (2) Suppose that $x \mid y^n$. If $n \leq m$, then also $x \mid y^m$; so $x \mid y$ by hypothesis. If $n > m$, then also $x \mid y^{km}$ with $k < n$. Thus $x \mid y^k$ by hypothesis, and hence $x \mid y$ by induction on n.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (4) Let I be a proper ideal of R. Suppose that $x^n = i \in I$ for some $x \in R$ and integer $n \geq 1$. Thus $x^{2n} = i^2$, and hence $i \mid x$ by hypothesis. Thus $x \in I$; so I is a radical ideal of R.

(4) $\Rightarrow$ (5) Clear.

(5) $\Rightarrow$ (6) Let $x \in R$. Then (x^2) is a radical ideal of R by hypothesis. Since $x^2 \in (x^2)$, we have $x \in (x^2)$, and hence $x = x^2 y$ for some $y \in R$. Thus R is von Neumann regular.

(6) $\Rightarrow$ (1) Let $x, y \in R$ with $x \mid y^m$; say $y^m = xd$ with $d \in R$. Since R is von Neumann regular, $y = ue$ with $u \in U(R)$ and $e \in R$ idempotent.[6, Corollary 3.3] Thus $y = ue = ue^m = u^{1-m}(ue)^m = u^{1-m}y^m = (u^{1-m}d)x$; so $x \mid y$. $\square$

Proposition 1.1 yields our next result on when $T(R)$ is von Neumann regular (also, see Theorem 2.3). In a similar manner, one may obtain criteria for $T(R)$ to be either strongly root closed or strongly (2,3)-closed (see Sec. 4 for the definitions), zero-dimensional (see Sec. 2), or seminormal (see Sec. 5).

Proposition 1.2. *The following statements are equivalent for a commutative ring R.*

(1) *$T(R)$ is von Neumann regular.*
(2) *Let $x, y \in R$. If $x \mid y^n$ for some integer $n \geq 1$, then $x \mid sy$ for some regular element $s \in R$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $T(R)$ is von Neumann regular and $x \mid y^n$ in R. By Proposition 1.1, $x \mid y$ in $T(R)$; so $y = zx$ for some $z = d/s \in T(R)$. Then $x \mid sy$ in R with s a regular element of R.

(2) $\Rightarrow$ (1) By Proposition 1.1, we need only show that if $x \mid y^n$ in $T(R)$ for some integer $n \geq 1$ and $x, y \in T(R)$, then $x \mid y$ in $T(R)$. Write $x = a/c$ and $y = b/d$. Then $x \mid y^n$ yields that $d^n ae = cfb^n$ for some $e \in R$ and regular element $f \in R$. Thus $a|(cfb)^n$ in R. By hypothesis, $a|s(cfb)$ in R for some regular element $s \in R$. Hence $x \mid y$ in $T(R)$. $\square$

By restricting the divisibility conditions in Proposition 1.1 to regular elements, we obtain several equivalent conditions for R to be a total quotient ring (i.e., $R = T(R)$, equivalently, each regular element of R is a unit of R). However, we lose the fact that R is reduced.

Proposition 1.3. *The following statements are equivalent for a commutative ring R.*

(1) *Let $x, y \in R$ with x a regular element of R. If $x \mid y^n$ for some integer $n \geq 1$, then $x \mid y$.*
(2) *Let $x, y \in R$ with y a regular element of R. If $x \mid y^n$ for some integer $n \geq 1$, then $x \mid y$.*
(3) *Let $x, y \in R$ with x, y regular elements of R. If $x \mid y^n$ for some integer $n \geq 1$, then $x \mid y$.*
(4) $R = T(R)$.

Proof. We first show that any of conditions (1)–(3) implies condition (4). Let $x \in R$ be a regular element of R. Then $x^2 \mid x^2$ implies that $x^2 \mid x$, and hence x is a unit of R. Thus condition (4) holds.

Conversely, suppose that condition (4) holds and that $x \mid y^n$. If either x or y is a regular element of R, then x is a unit of R by hypothesis, and hence $x \mid y$. Thus conditions (1)–(3) all hold. $\square$

Example 1.4. The ring $\mathbb{Z}/4\mathbb{Z}$ satisfies the equivalent conditions of Proposition 1.3, but not those of Proposition 1.1. Note that $\mathbb{Z}/4\mathbb{Z}$ is a total quotient ring, but it is not reduced.

2 ZERO-DIMENSIONAL TOTAL QUOTIENT RINGS

In this section, we give several conditions on R for $T(R)$ to be either zero-dimensional or von Neumann regular. We then show that a condition introduced in Ref. [1], and further used in Ref. [2], on a subring R of a direct product of integral domains is equivalent to $T(R)$ being von Neumann regular.

Recall that a commutative ring R is called *π-regular* if for each $x \in R$, there is a $y \in R$ and an integer $n \geq 1$ such that $x^{2n}y = x^n$, i.e., $x^{2n} \mid x^n$. Then R is π-regular if and only if R is zero-dimensional.[6, Theorem 3.1] Thus, in the spirit of Proposition 1.2, one may easily show that $T(R)$ is zero-dimensional if and only if for each $x \in R$, there is an integer $n \geq 1$ and a regular element $s \in R$ such that $x^{2n} \mid sx^n$ in R. We give a much more interesting internal characterization of when $T(R)$ is zero-dimensional in Theorem 2.2, but first a lemma.

Lemma 2.1. *Let R be a commutative ring and $x, y \in R$.*

(1) *Suppose that $xy \in nil(R)$ and let $n \geq 1$ be an integer. Then $x + y$ is a regular element of R if and only if $x^n + y^n$ is a regular element of R.*
(2) *Suppose that $xy = 0$. If $ax + by$ is a regular element of R for some $a, b \in R$, then $x + y$ is also a regular element of R. (Thus the ideal (x, y) contains a regular element of R if and only if $x + y$ is a regular element of R.)*
(3) *Suppose that $xy = 0$. Then $x + y$ is a regular element of R if and only if $x^m + y^n$ is a regular element of R for some integers $m, n \geq 1$.*

Proof. (1) Let $xy \in nil(R)$. By the Binomial Theorem, $(x + y)^n = x^n + y^n + z$ with $z \in nil(R)$. Thus $(x + y)^n$ is a unit in $T(R)$ if and only if $x^n + y^n$ is a unit in $T(R)$. Hence $x + y$ is a regular element of R if and only if $x^n + y^n$ is a regular element of R.

(2) Suppose that $xy = 0, ax + by$ is a regular element of R, and that $(x + y)d = 0$ for some $0 \neq d \in R$. If $yd = 0$, then also $xd = 0$; and hence $(ax + by)d = 0$, a contradiction. Thus we may assume that $yd \neq 0$. Then $(x + y)yd = 0$ yields $y^2d = 0$ since $xy = 0$. Hence $(ax + by)yd = 0$, a contradiction. Thus $x + y$ is a regular element of R.

(3) This follows easily from part (2) above. □

Theorem 2.2. *The following statements are equivalent for a commutative ring R.*

(1) *$T(R)$ is zero-dimensional.*
(2) *For each $x \in R$, there is a $y \in R$ and an integer $n \geq 1$ such that $x^n y = 0$ and $x^n + y$ is a regular element of R.*
(3) *For each $x \in R$, there is a $y \in R$ such that $xy \in nil(R)$ and $x + y$ is a regular element of R.*

Proof. (1) $\Rightarrow$ (2) Suppose that $T(R)$ is zero-dimensional, and let $x \in R$. Since $T(R)$ is π-regular,[6, Theorem 3.1] there is a $z/s \in T(R)$ and an integer $n \geq 1$ such that $x^{2n}(z/s) = x^n$. Thus $x^n(s - x^n z) = 0$. Let $y = s - x^n z$. Then $x^n y = 0$ and $zx^n + y = s$ is a regular element of R. Hence $x^n + y$ is a regular element of R by Lemma 2.1(2).

(2) $\Rightarrow$ (3) Suppose that R satisfies condition (2), and let $x \in R$. Then there is a $y \in R$ and an integer $n \geq 1$ such that $x^n y = 0$ and $x^n + y$ is a regular element of R. Thus $xy \in nil(R)$. Since $x^n y = 0$ and $x^n + y$ is a regular element of R, also $(x^n)^n + y^n = (x^{n^2 - n})x^n + y^n$ is a regular element of R by Lemma 2.1(1). Hence $x^n + y^n$ is a regular element of R by Lemma 2.1(2), and thus $x + y$ is a regular element of R by Lemma 2.1(1) again.

(3) $\Rightarrow$ (1) Suppose that R satisfies condition (3). To show that $T(R)$ is zero-dimensional, it is sufficient to show that each non-minimal prime ideal

Q of R contains a regular element of R. Let $P \subset Q$ be distinct prime ideals of R, and choose an $x \in Q - P$. By hypothesis, there is a $y \in R$ such that $xy \in nil(R)$ and $x + y$ is a regular element of R. Thus $y \in P \subset Q$, and hence $x + y \in Q$. Thus Q contains a regular element of R. □

In the next result, a slight modification of the conditions in Theorem 2.2 forces R to be reduced, and hence von Neumann regular.

Theorem 2.3. *The following statements are equivalent for a commutative ring R.*

(1) *$T(R)$ is von Neumann regular.*
(2) *For each $x \in R$, there is a $y \in R$ such that $xy = 0$ and $x + y$ is a regular element of R.*

Proof. (1) $\Rightarrow$ (2) Suppose that $T(R)$ is von Neumann regular. Then $T(R)$, and hence R, is reduced. Thus (2) follows from Theorem 2.2 since $nil(R) = \{0\}$.

(2) $\Rightarrow$ (1) Suppose that R satisfies condition (2). We first show that R is reduced. Let $x \in nil(R)$; say $x^n = 0$ for some integer $n \geq 1$. Then by hypothesis, there is a $y \in R$ such that $xy = 0$ and $x + y$ is a regular element of R. Thus $y^n = x^n + y^n$ is a regular element of R by Lemma 2.1(1). Hence y is also a regular element of R, so $x = 0$. Thus R, and hence $T(R)$, is reduced. It follows from Theorem 2.2 that $T(R)$ is zero-dimensional, and hence von Neumann regular.[6, Theorem 3.1] □

Remark 2.4.

(1) Theorems 2.2 and 2.3 also follow from Ref. [6, Theorem 3.2 and Corollary 3.3] via Lemma 2.1 and (for Theorem 2.3) the observation in the proof of (2) $\Rightarrow$ (1) of Theorem 2.3 that R is reduced. The above two results in Ref. [6] are from Ref. [9].
(2) There are many other characterizations in terms of R of when $T(R)$ is von Neumann regular. For example, see Ref. [6, Theorem 4.5] for conditions concerning when $\mathrm{Min}(R)$, the set of minimal prime ideals of R, is compact.

We end this section by showing that condition (2) in Theorem 2.3 is equivalent to a concept introduced in Ref. [1], and further used in Ref. [2], for studying subrings of a direct product of integral domains.

Let R be a subring of the direct product ΠR_α of a family $\{R_\alpha\}$ of integral domains. As in Ref. [1], for $x = (x_\alpha), y = (y_\alpha) \in R \subset \Pi R_\alpha$, we say that *$y$ extends x*, written yEx, if $y_\alpha = x_\alpha$ whenever $x_\alpha \neq 0$. We say that x *extends to a regular element of* R if there is a regular element $y \in R$ such that

yEx. Note that for $x, y \in R$, we have $xy = 0$ if and only if $x + y$ extends x. Thus y extends x if and only if $x(y - x) = 0$.

For any commutative ring S, define an ordering on S by $a \leq b$ if either $a = 0$ or $a = b$. For a family $\{R_\alpha\}$ of commutative rings, the induced product order on ΠR_α is then $(x_\alpha) \leq (y_\alpha) \Leftrightarrow$ either $x_\alpha = 0$ or $x_\alpha = y_\alpha$ for each α, i.e., (y_α) extends (x_α) in ΠR_α. This ordering restricts to the ordering E defined above on any subring R of ΠR_α.

Recall that a commutative ring R is reduced if and only if R is a subring of the direct product ΠR_α of some family $\{R_\alpha\}$ of integral domains, and that R is a subring of the product of a finite number of integral domains if and only if R is reduced with only a finite number of minimal prime ideals. Thus much of our earlier work in Refs. [1,2] for reduced rings was set in the context of subrings of a direct product of integral domains, and the concept of "extending to a regular element" played a key role. Our next result shows that the concept of "extending to a regular element" is independent of the embedding of the reduced ring R in a direct product of integral domains. Corollary 2.6 then relates this concept to $T(R)$ being von Neumann regular.

Proposition 2.5. *Let R be a subring of a direct product of integral domains. Then $x \in R$ extends to a regular element of R if and only if there is a $y \in R$ such that $xy = 0$ and $x + y$ is a regular element of R.*

Proof. ($\Rightarrow$) Let $x \in R$ and zEx with $z \in R$ regular. Let $y = z - x$. Then $xy = 0$ and $x + y = z$ is a regular element of R.

($\Leftarrow$) Let $x \in R$. By hypothesis, there is a $y \in R$ such that $xy = 0$ and $x + y$ is a regular element of R. Clearly $x + y$ extends x since $xy = 0$. $\square$

Corollary 2.6. *Let R be a commutative ring which is a subring of a direct product of integral domains. Then each $x \in R$ extends to a regular element of R if and only if $T(R)$ is von Neumann regular.*

Proof. This follows immediately from Theorem 2.3 and Proposition 2.5.

Thus, in several results in Refs. [1,2], the hypothesis "every element of R can be extended to a regular element of R" may be replaced by "$T(R)$ is von Neumann regular" (also, see Sec. 4 and 5).

The next corollary generalizes the well-known fact that a reduced commutative ring with only a finite number of minimal prime ideals (in particular, a reduced commutative Noetherian ring) has von Neumann regular total quotient ring.

Corollary 2.7. *Let R be a reduced commutative ring such that each nonzero zerodivisor of R is contained in only a finite number of minimal prime ideals of R. Then $T(R)$ is von Neumann regular.*

Proof. We may view R as a subring of $\Pi(R/P_\alpha)$, where $\{P_\alpha\}$ is the set of minimal prime ideals of R. By hypothesis, each $0 \neq (r_\alpha) \in Z(R)$ has only a finite number of zero entries, and hence can be extended to a regular element of R by an argument similar to that in the proof of Ref. [1, Lemma 2.5]. Thus $T(R)$ is von Neumann regular by Corollary 2.6. □

3 STRONGLY COMPLETELY INTEGRALLY CLOSED RINGS

Let R be a commutative ring. Recall that $x \in T(R)$ is *almost integral over* R if there is a regular element $s \in R$ such that $sx^n \in R$ for all integers $n \geq 1$. As usual, R is called *completely integrally closed* (CIC) if whenever $x \in T(R)$ is almost integral over R, then $x \in R$ (equivalently, if $b^n | sa^n$ for all integers $n \geq 1$ with $a \in R$ and $b, s \in R$ regular elements of R, then $b|a$). In the spirit of Ref. [2], we define R to be *strongly completely integrally closed* (SCIC) if whenever $b^n | sa^n$ for all integers $n \geq 1$ with $a, b \in R$ and $s \in R$ a regular element of R, then $b|a$. (Here we have replaced $a/b \in R$, where $a \in R, b \in R - Z(R)$, by $b|a$ in R, and we allow $b \in Z(R)$.) Clearly, if a ring R is SCIC, then R is also CIC (but not conversely, see Example 3.6).

Our first result gives a trivial case when R is always SCIC.

Proposition 3.1. *Let R be a commutative ring. If $R = T(R)$, then R is SCIC. In particular, a zero-dimensional commutative ring is SCIC.*

Proof. The first part is clear. For the second part, just recall that $R = T(R)$ when R is zero-dimensional. □

As in Ref. [10], a commutative ring R is called *additively regular* if for each $x \in T(R)$, there is a $y \in R$ such that $x + y$ is a regular element (unit) of $T(R)$; equivalently, R is additively regular if and only if for all $x, y \in R$ with y a regular element of R, there is an $a \in R$ such that $x + ay$ is a regular element of R. If either $T(R)$ is zero-dimensional or $Z(R)$ is a finite union of prime ideals, then R is additively regular.[6, Theorems 7.4 and 7.2] In particular, von Neumann regular rings, Noetherian rings, and reduced rings with only a finite number of minimal prime ideals are additively regular.

We define R to be *strongly additively regular* if for each $x \in R$, there is a $y \in R$ such that $xy = 0$ and $x + y$ is a regular element of R. Of course, by Theorem 2.3, R is strongly additively regular if and only if $T(R)$ is von Neumann regular. Thus a strongly additively regular ring is additively regular (this was also observed in a different context in Ref. [2, Proposition 2.5]). However, the converse is false since any zero-dimensional ring is additively regular; for a reduced example, see Example 3.7.

We next show that a strongly additively regular commutative ring R is SCIC if and only if it is CIC (i.e., when $T(R)$ is von Neumann regular). Example 3.6 shows that it is not enough to just assume that R is additively regular.

Proposition 3.2. *Let R be a strongly additively regular commutative ring (equivalently, $T(R)$ is von Neumann regular). Then R is SCIC if and only if R is CIC.*

Proof. We need only show that R is SCIC if R is CIC. Suppose that R is CIC, and let $a, b, s \in R$ with s a regular element of R such that $b^n | sa^n$ for all integers $n \geq 1$; say $d_n b^n = sa^n$ with each $d_n \in R$. Since R is strongly additively regular, there is a $z \in R$ such that $bz = 0$ and $b + z$ is a regular element of R. Then also $az = 0$ since s is regular and R is reduced by the proof of Theorem 2.3. Thus $(d_n b)(b+z)^n = d_n b^{n+1} = bsa^n = [(b+z)s]a^n$ for all integers $n \geq 1$. Hence $(b+z)^n | [(b+z)s]a^n$ for all integers $n \geq 1$ with $b+z$, $(b+z)s \in R$ regular. Thus $a = (b+z)w$ for some $w \in R$ since R is CIC. Note that $az = bz = 0$ yields $wz^2 = 0$; so $wz = 0$ since R is reduced. Hence $a = bw$, and thus R is SCIC. □

Corollary 3.3. *Let R be a reduced commutative ring such that each nonzero zerodivisor of R is contained in only a finite number of minimal prime ideals of R. Then R is SCIC if and only if R is CIC. In particular, this holds if R is a reduced commutative Noetherian ring.*

Proof. This follows directly from Proposition 3.2 and Corollary 2.7. □

Let M be an R-module. Then the *idealization* of R and M is the ring $R(+)M$ with underlying set $R \times M$ under coordinatewise addition, and multiplication given by $(r, x)(s, y) = (rs, ry + sx)$. In the next several results, we will use the facts that $U(R(+)M) = U(R)(+)M$ ([6, Theorem 25.1(6)]) and $Z(R(+)M) = A(+)M$, where $A = Z(R) \cup Z(M)$ ([6, Theorem 25.3]). See Ref. [6, Sec. 25] for more details on idealization. We next determine when certain idealizations are SCIC.

Proposition 3.4. *Let R be a subring of a commutative ring B such that each regular element of R is also a regular element of B (for example, if $B \subset T(R)$). Then $R(+)B$ is SCIC if and only if $R = T(R)$.*

Proof. Let $A = R(+)B$. First suppose that $R = T(R)$. Then each element of R is either a unit or a zerodivisor, and hence each element of A is either a unit or a zerodivisor. Thus $A = T(A)$, and hence A is SCIC by Proposition 3.1. Conversely, suppose that $R \neq T(R)$. Let $x \in R$ be a nonunit regular element. Define $s = (x, 0)$, $a = (0, 1)$, and $b = (0, x)$ in A. Then s is a regular element of A and $b^n \mid sa^n$ in A for all integers $n \geq 1$. However, $b \nmid a$ in A since x is not a unit of R. Thus A is not SCIC. $\square$

Proposition 3.5. *Let R be a CIC commutative ring with $R \neq T(R)$. Then $R(+)T(R)$ is CIC, but not SCIC.*

Proof. By Proposition 3.4, $A = R(+)T(R)$ is not SCIC. We next show that A is CIC. Suppose that $(b, y)^n \mid (s, z)(a, x)^n$ for all integers $n \geq 1$ and $(a, x), (b, y), (s, z) \in A$ with $(b, y), (s, z)$ regular elements of A. Then $b^n \mid sa^n$ in R for all integers $n \geq 1$ with $b, s \in R$ regular. Thus $a = bc$ for some $c \in R$ since R is CIC. Hence $(a, x) = (b, y)(c, w)$ in A with $w = (x - cy)/b \in T(R)$. Thus A is CIC. $\square$

Example 3.6. By Proposition 3.5, $R = \mathbb{Z}(+)\mathbb{Q}$ is CIC, but not SCIC. Note that R is additively regular, but not strongly additively regular since R is not reduced.

Example 3.7. (A reduced commutative ring R which is additively regular, but not strongly additively regular.) Let K be a field, and let $A = K[X_1, X_2, \ldots] = K[\{X_n\} \mid n \in \mathbb{N}\}]$. Then $I = (\{X_iX_j \mid i, j \in \mathbb{N}, i \neq j\})$ is a radical ideal of A contained in the maximal ideal $N = (\{X_n \mid n \in \mathbb{N}\})$. Thus $R = A_N/I_N$ is a quasilocal reduced ring with maximal ideal $M = N_N/I_N$. Clearly $Z(R) = M$, and thus $T(R) = R$. Hence R is additively regular. However, for any $0 \neq x \in M$, there is no $y \in R$ such that $xy = 0$ and $x + y$ is a regular element of R. Thus R is not strongly additively regular.

Question 3.8. Is there a reduced commutative ring R which is CIC, but not SCIC?

4 CLOSEDNESS WITH RESPECT TO UNITS OF $T(R)$

In this section, we continue our investigation from Refs. [1,2] of when R satisfies certain "closedness" properties with respect to units of $T(R)$, i.e., if $x \in U(T(R))$ (equivalently, $x = a/b$ with a, b regular elements of R) satisfies a given closedness property, then $x \in R$ (equivalently, $b \mid a$ in R). The "closedness" properties we consider here are completely integrally closed, integrally closed, root closed, and (2,3)-closed. It is clear that these four "closedness" properties with respect to units of $T(R)$ are inherited by direct products, intersections of overrings in $T(R)$, and (except for the CIC case) localizations in $T(R)$. We also briefly discuss rings such that $T(R) - R \subset U(T(R))$.

Recall that a commutative ring R is *root closed* (resp., $(2, 3)$-closed) if whenever $x^n \in R$ for some integer $n \geq 1$ (resp., $x^2, x^3 \in R$) and $x \in T(R)$, then $x \in R$. As in Ref. [2], we say that R is *strongly root closed* (resp., *strongly (2,3)-closed*) if whenever $b^n \mid a^n$ for some integer $n \geq 1$ (resp., $b^2 \mid a^2, b^3 \mid a^3$) and $a, b \in R$, then $b \mid a$ in R. (Here we have replaced $a/b \in R$, where $a \in R$ and $b \in R - Z(R)$, by $b \mid a$ in R, and we allow $b \in Z(R)$.) Note that R is strongly $(2,3)$-closed (resp., $(2,3)$-closed) if and only if whenever $b^n \mid a^n$ for all sufficiently large integers $n \geq 1$ and $a, b \in R$ (resp., $a \in R$ and $b \in R - Z(R)$), then $b \mid a$ in R. Clearly a strongly root closed (resp., strongly $(2,3)$-closed) ring is root closed (resp., $(2,3)$-closed). Examples in Ref. [2] show that the converse is false.

Our first result is the CIC analog of Ref. [2, Propositions 2.2–2.4], i.e., if R is root closed (resp., $(2,3)$-closed, integrally closed) with respect to units of $T(R)$ and R is additively regular (in particular, if $T(R)$ is von Neumann regular), then R is root closed (resp., $(2,3)$-closed, integrally closed). A stronger result is given in Theorem 4.2.

Proposition 4.1. *The following statements are equivalent for an additively regular commutative ring R.*

(1) *R is completely integrally closed.*
(2) *If $x \in U(T(R))$ is almost integral over R, then $x \in R$.*

Proof. Clearly (1) $\Rightarrow$ (2). Conversely, suppose that condition (2) holds, and let $x \in T(R)$ be almost integral over R. Then there is a regular element $s \in R$ such that $sx^n \in R$ for all integers $n \geq 1$. Since R is additively regular, there is a $y \in R$ such that $x + y \in U(T(R))$. Then $s(x+y)^n \in R$ for all integers $n \geq 1$. Thus $x + y \in R$ by hypothesis, and hence $x \in R$. Thus R is completely integrally closed. □

In the spirit of Refs. [1,2], and Sec. 3, we consider the following four conditions related to root closedness on a commutative ring R with $a, b \in R$ and $n \geq 1$ an integer. (Similar conditions can be given for $(2,3)$-closedness, integral closedness, and complete integral closedness; we leave the specific details to the interested reader.)

(1) If $b^n \mid a^n$, then $b \mid a$ (i.e., R is strongly root closed).
(2) If $b^n \mid a^n$ with b regular, then $b \mid a$ (i.e., R is root closed).
(3) If $b^n \mid a^n$ with a, b regular, then $b \mid a$ (i.e., R is root closed with respect to units of $T(R)$).
(4) If $b^n \mid a^n$ with a regular, then $b \mid a$.

Clearly conditions (1) $\Rightarrow$ (2) $\Rightarrow$ (3), and conditions (3) $\Leftrightarrow$ (4). By Ref. [2, Example 3.1], condition (2) does not imply condition (1).

We always have: R is CIC wrt units of $T(R) \Rightarrow R$ is integrally closed wrt units of $T(R) \Rightarrow R$ is root closed wrt units of $T(R) \Rightarrow R$ is $(2,3)$-closed wrt units of $T(R)$. Although a CIC ring is always root closed, an SCIC ring need not be strongly root closed (cf. Proposition 3.1).

In Refs. [2, Question 2.7], we asked if R is integrally closed (resp., $(2,3)$-closed, root closed) with respect to units of $T(R)$ implies that R is integrally closed (resp., $(2,3)$-closed, root closed). In Refs. [2, Propositions 2.2–2.4], we showed that this is true if R is additively regular. We next generalize this to the class of Marot rings. As in Ref. [6], a commutative ring R is called a *Marot ring* if every ideal of R which contains a regular element is generated by regular elements. Examples of Marot rings include integral domains, Noetherian rings, rings such that $Z(R)$ is a finite union of prime ideals, polynomial rings, and rings with zero-dimensional total quotient ring.[6, Theorems 7.2, 7.4, and 7.5] An additively regular ring is a Marot ring,[6, Theorem 7.2] but not conversely (see Refs. [6, Example 12, p. 185] or Ref. [11]).

Theorem 4.2. *Let R be a commutative Marot ring. Then R is root closed (resp., $(2,3)$-closed, integrally closed, completely integrally closed) if and only if R is root closed (resp., $(2,3)$-closed, integrally closed, completely integrally closed) with respect to units of $T(R)$.*

Proof. Suppose that R is a Marot ring which is root closed with respect to units of $T(R)$. Let $a, b \in R$ with b regular, and suppose that $(a/b)^n \in R$, i.e., $b^n \mid a^n$, for some integer $n \geq 1$. Let $I = (a, b^n)$, and let x be a regular element of I. Then $x = ca + db^n$ for some $c, d \in R$. Using the Binomial Theorem, one can easily show that $b^n \mid x^n$ since $b^n \mid a^n$, and hence $b \mid x$ by hypothesis. Since R is a Marot ring, a is a linear combination of regular elements of I, and thus $b \mid a$. Hence $a/b \in R$; so R is root closed. The converse is clear. The proofs for the other three closedness properties are similar. For the $(2,3)$-closed case, let $I = (a, b^3)$; for the integrally closed case, let $I = (a, b)$; and for the completely integrally closed case, let $I = (a, b)$. Details are left to the reader. □

We next give an example of a reduced commutative ring R which satisfies the four closedness conditions with respect to units of $T(R)$, but not with respect to $T(R)$. This answers questions raised in Ref. [2, Question 2.7] and Refs. [1, Sec. 3]. We would like to thank Thomas G. Lucas for suggesting this example.

Example 4.3. (A reduced commutative ring R which is root closed (resp., $(2,3)$-closed, integrally closed, completely integrally closed) with respect to units of $T(R)$, but R is not root closed (resp., $(2,3)$-closed, integrally closed, completely integrally closed).) We employ the "$A + B$" construction, see Refs. [6, Section 26] and [12] for more details. Let K be a field, $D = K[X^2, X^3, XY, Y]$, and let $\mathcal{P} = \{P \in \mathrm{Spec}(R) \mid htP = 1 \text{ and } Y \notin P\}$. Let $\mathcal{A}$ be an indexing set for $\mathcal{P}$, and let $I = \mathcal{A} \times \mathbb{N}$. For each $i = (\alpha, n) \in I$, let

$K_i = qf(D/P_\alpha)$, and let $B = \oplus_{i\in I} K_i$. Finally, let $R = D + B$. It is easiest to view R as $R = D \oplus B$ with coordinatewise addition and $(d_1, b_1)(d_2, b_2) = (d_1 d_2, d_1 b_2 + d_2 b_1 + b_1 b_2)$. Then R is reduced and one can easily check that $R - Z(R) = \{aY^n \mid 0 \neq a \in K \text{ and } n \geq 1\}$ (here we identify $f \in D$ with $(f, 0)$). One can then easily verify that R satisfies each of the four closedness conditions with respect to units of $T(R)$, but not with respect to $T(R)$. For example, $Y^2 \mid (XY)^2$ in R, but $Y \nmid XY$ in R. $\square$

Clearly a strongly $(2,3)$-closed ring is reduced, and $\mathrm{nil}(T(R)) = \mathrm{nil}(R)$ when R is $(2,3)$-closed. We next show that $\mathrm{nil}(T(R)) = \mathrm{nil}(R)$ when R is $(2,3)$-closed with respect to units of $T(R)$ (thus also $\mathrm{nil}(T(R)) = \mathrm{nil}(R)$ if R is root closed, integrally closed, or completely integrally closed with respect to units of $T(R)$).

Proposition 4.4. *If a commutative ring R is $(2,3)$-closed with respect to units of $T(R)$, then $\mathrm{nil}(T(R)) = \mathrm{nil}(R)$.*

Proof. Clearly $nil(R) \subset nil(T(R))$. Conversely, suppose that $(x/y)^n = 0$ for some integer $n \geq 1$. Then $(x + y^n)/y = (x/y) + y^{n-1}$ is a unit in $T(R)$ with $[(x + y^n)/y]^m \in R$ for all integers $m \geq n$. Thus $(x + y^n)/y \in R$ by hypothesis, and hence $x/y \in R$. $\square$

Corollary 4.5. *Let R be a commutative ring.*

(1) *If R is $(2,3)$-closed with respect to units of $T(R)$, then $\mathrm{nil}(R) \subset sR$ for each regular element $s \in R$.*
(2) *Suppose that $nil(R) = Z(R)$. Then R is $(2,3)$-closed (resp., root closed, integrally closed, completely integrally closed) if and only if R is $(2,3)$-closed (resp., root closed, integrally closed, completely integrally closed) with respect to units of $T(R)$.*

Proof. (1) This follows directly from Proposition 4.4.

(2) We show that if R is $(2,3)$-closed with respect to units of $T(R)$ and $\mathrm{nil}(R) = Z(R)$, then R is $(2,3)$-closed; the proofs of the other cases are left to the reader. Let $x = a/b \in T(R)$ with $x^2, x^3 \in R$. If $a \in R - Z(R)$, then $x \in R$ by hypothesis. If $a \in Z(R) = \mathrm{nil}(R)$, then $x \in \mathrm{nil}(T(R)) = \mathrm{nil}(R) \subset R$. Thus R is $(2,3)$-closed. $\square$

If $T(R) - R \subset U(T(R))$, then clearly R is $(2,3)$-closed (resp., root closed, integrally closed, completely integrally closed) if and only if R is $(2,3)$-closed (resp., root closed, integrally closed, completely integrally closed) with respect to units of $T(R)$. Thus it is of interest to characterize the commutative rings R such that $T(R) - R \subset U(T(R))$. Recall that an ideal of a commutative ring R is said to be *divided* if it is comparable to every other ideal of R (equivalently, to *every* principal ideal of R). Note that if $Z(R)$ is a divided ideal (and hence necessarily prime), then $T(R) - R \subset U(T(R))$. We next show that the converse is also true except in trivial cases (note that if $R = T(R)$, then $Z(R) = R - U(R)$; so, in this case, $Z(R)$ is an ideal of R if and only if R is quasilocal with maximal ideal $Z(R)$).

Proposition 4.6. *Let R be a commutative ring which contains a nonunit regular element (i.e., $R \neq T(R)$). Then $T(R) - R \subset U(T(R))$ if and only if $Z(R)$ is a divided prime ideal of R.*

Proof. We have already observed that $T(R) - R \subset U(T(R))$ if $Z(R)$ is a divided (prime) ideal of R. Conversely, suppose that R contains a nonunit regular element and $T(R) - R \subset U(T(R))$. Thus $Z(R) \subset sR$ for each regular element s of R. We need only show that $Z(R)$ is an ideal of R. Let $I = \cap\{xR \mid x$ is a regular element of $R\}$. By hypothesis, I is a proper ideal of R containing $Z(R)$. We show that $I = Z(R)$. If not, then there is an $x \in I - Z(R)$. Thus $Z(R) \subset xR \subset I \subset xR$, and hence $I = xR$. Similarly, $I = x^2R$. Thus $x \in U(R)$, a contradiction. □

We next consider several classes of commutative rings R such that $Z(R)$ is a divided prime ideal of R. Note that such rings are Marot rings. As in Ref. [4], we say that a commutative ring R is a *pseudo-valuation ring* (PVR) if aP and bR are comparable for all $a, b \in R$ and prime ideals P of R. A PVR is necessarily quasilocal (for this and other results about PVRs, see Refs. [3,4]). We say that a commutative ring R is a Φ-pseudo-valuation ring (Φ-PVR) if *nil*(R) is a divided prime ideal of R and for each prime ideal $P \neq \mathrm{nil}(R)$ of R, aP and bR are comparable for all $a, b \in R - nil(R)$ (cf.[13, Corollary 7] and [14, Proposition 1.1(6)]); equivalently, $\mathrm{nil}(R)$ is a divided prime ideal of R and $R/nil(R)$ is a PVR. [15, Proposition 2.9, 16, Theorem 3.1] Also, see[16] for some other generalizations of PVRs.

Corollary 4.7. *Let R be a commutative ring such that $Z(R)$ is a divided prime ideal of R. Then R is $(2, 3)$-closed (resp., root closed, integrally closed, completely integrally closed) if and only if R is $(2, 3)$-closed (resp., root closed, integrally closed, completely integrally closed) with respect to units of $T(R)$. In particular, the above statement holds for a PVR or a Φ-PVR.*

Proof. The first statement is clear since in this case $T(R) - R \subset U(T(R))$ by Proposition 4.6. For the "in particular" statement, just note that a PVR or a Φ-PVR clearly satisfies the given hypothesis. □

5 SEMINORMAL RINGS

Following Swan,[17] we say that a (necessarily reduced) commutative ring R is *seminormal* if whenever $a^2 = b^3$ for $a, b \in R$, then $a = c^3$ and $b = c^2$

for some $c \in R$. The importance of seminormality is that $\mathrm{Pic}(R[X]) = \mathrm{Pic}(R)$ if and only if $R/\mathrm{nil}(R)$ is seminormal.[17, Theorem 1] In Ref. [1], we studied several variants of seminormality. In this section, we briefly continue that investigation. The following two conditions were conditions (2) and (6) in Ref. [1], respectively.

(5) If $a^2 = b^3$ with $a, b \in R$, then $b \mid a$ in R.
(6) If $a^2 = b^3$ with $a, b \in R$ regular, then $b \mid a$ in R.

Clearly condition (5) implies condition (6) and any seminormal ring satisfies conditions (5) and (6). However, condition (6) is not equivalent to R being seminormal since any total quotient ring satisfies (6). The ring R in Example 4.3 also satisfies (6), but R is not seminormal. In Ref. [1, Example 2.7(a)], an example was given of a reduced commutative ring which satisfies condition (5), but is not seminormal. In fact, R is seminormal if and only if R is $(2, 3)$-closed and $T(R)$ is seminormal.[1, Theorem 3.1(a)] In the spirit of Proposition 1.2, one may easily show that $T(R)$ is seminormal if and only if whenever $a^2 = b^3$ for $a, b \in R$, there are $c, s \in R$ with s regular such that $s^3a = c^3$ and $s^2b = c^2$.

We next give several other conditions equivalent to condition (6).

The $(2) \Leftrightarrow (3)$ equivalence of Proposition 5.1 is also in Ref. [1, Theorem 3.1(c)].

Proposition 5.1. *The following statements are equivalent for a commutative ring R.*

(1) *If $a^2 = b^3$ with $a, b \in R$ regular, then $a = c^3$ and $b = c^2$ for some (regular) $c \in R$ (i.e., R is seminormal with respect to regular elements of R).*
(2) *If $a^2 = b^3$ with $a, b \in R$ regular, then $b \mid a$ (i.e., R satisfies condition (6)).*
(3) *If $b^2 \mid a^2$ and $b^3 \mid a^3$ with $a, b \in R$ regular, then $b \mid a$ (i.e., R is $(2, 3)$-closed with respect to units of $T(R)$).*

Proof. $(1) \Rightarrow (2)$ Clear.

$(2) \Rightarrow (3)$ Suppose that $b^2 \mid a^2$ and $b^3 \mid a^3$ with $a, b \in R$ regular. Then $\beta = a^2/b^2$ and $\alpha = a^3/b^3$ are regular elements of R which satisfy $\alpha^2 = \beta^3$, and hence $\beta \mid \alpha$ by hypothesis. Thus $b \mid a$.

$(3) \Rightarrow (1)$ Suppose that $a^2 = b^3$ with $a, b \in R$ regular. Then $b^2 \mid a^2$ and $b^3 \mid a^3$, and hence $b \mid a$ by hypothesis. Let $c = a/b \in R$. Then $a = c^3$ and $b = c^2$. $\square$

Our next result is a slight generalization of Ref. [1, Theorem 3.3] via Corollary 2.6. As a special case, if $T(R)$ is von Neumann regular, then R is seminormal if and only if R satisfies any of the three equivalent conditions in Proposition 5.1 (recall that a von Neumann regular ring is seminormal).

Proposition 5.2. *Let R be a commutative Marot ring such that $T(R)$ is seminormal. Then R is seminormal if and only if R satisfies condition* (6).

Proof. By Theorem 4.2 and Proposition 5.1 R satisfies condition (6) if and only if R is $(2,3)$-closed. As mentioned above, R is seminormal if and only if $T(R)$ is seminormal and R is $(2,3)$-closed.[1, Theorem 3.1(a)] The result follows. □

Corollary 5.3. (cf.[1, Theorem 3.3]) *Let R be a strongly regular commutative ring* (*i.e., $T(R)$ is von Neuman regular*). *Then R is seminormal if and only if R satisfies condition* (6). □

REFERENCES

1. Anderson, D.F.; Badawi, A. Conditions Equivalent to Seminormality in Certain Classes of Commutative Rings. *Lecture Notes in Pure and Applied Mathematic*; Marcel Dekker: New York, Basel, 2001; Vol. 220, 49–59.
2. Anderson, D.F.; Badawi, A. On Root Closure in Commutative Rings. Arabian J. Sci. Engrg. **2001**, *26* (*IC*), 17–30.
3. Anderson, D.F; Badawi, A.; Dobbs, D.E. Pseudo-Valuation Rings, II. Boll. Un. Mat. Ital. **B 2000**, *3* (8), 535–545.
4. Badawi, A.; Anderson, D.F.; Dobbs, D.E. Pseudo-Valuation Rings. *Lecture Notes in Pure and Applied Mathematics*; Marcel Dekker: New York, Basel, 1997; Vol. 185, 56–67.
5. Gilmer, R. *Multiplicative Ideal Theory*; Marcel Dekker: New York, Basel, 1972.
6. Huckaba, J.A. *Commutative Rings with Zero Divisors*; Marcel Dekker: New York, Basel, 1988.
7. Kaplansky, I. *Commutative Rings*; Rev. Ed., University of Chicago Press: Chicago, 1974.
8. Gilmer, R. Zero-Dimensional Rings. *Lecture Notes in Pure and Applied Mathematics*; Marcel Dekker: New York, Basel, 1995; Vol. 171, 1—52.
9. Arapovic, M. Characterizations of the 0-Dimensional Rings. Glas. Mat. Ser. III **1983**, *18*, 39—46.
10. Gilmer, R.; Huckaba, J.A. Δ-Rings. J. Algebra **1974**, *28*, 414–432.
11. Matsuda, R. On Marot Rings. Proc. Japan Acad. Ser. A Math. Sci. **1984**, *60*, 134—137.

12. Lucas, T.G. Root Closure and R[X]. Comm. Algebra **1989**, *17*, 2393–2414.
13. Badawi, A. On Φ-Pseudo-Valuation Rings. *Lecture Notes in Pure and Applied Mathematics*; Marcel Dekker: New York, Basel, 1999; Vol. 205, 101–110.
14. Badawi, A. On Φ-Pseudo-Valuation Rings. II, Houston J. Math. **2000**, *26*, 473–480.
15. Badawi, A. On Divided Rings. and Φ-Pseudo-Valuation Rings, Internat. J. Commutative Rings, (to appear).
16. Chang, G.W. Generalizations of Pseudo-Valuation Rings. Internat. J. Commutative Rings, (to appear).
17. Swan, R.G. On Seminormality. J. Algebra **1980**, *67*, 210–229.

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